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Reflexivity of the Toeplitz operators on the Hardy space on the upper half–plane

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all linear and bounded operators on $\mathcal{H}$. A *reflexive closure* of a subspace $\mathcal{S} \subset \mathcal{B}(\mathcal{H})$ is given by

$$\operatorname{ref}\mathcal{S} = \{B \in \mathcal{B}(\mathcal{H}) : Bh \in \overline{\mathcal{S}h} \text{ for all } h \in \mathcal{H}\}.$$

It is clear that $\mathcal{S} \subset \operatorname{ref}\mathcal{S} \subset \mathcal{B}(\mathcal{H})$. The subspace $\mathcal{S}$ is said to be *reflexive*, if $\operatorname{ref}\mathcal{S} = \mathcal{S}$ and *transitive*, if $\operatorname{ref}\mathcal{S} = \mathcal{B}(\mathcal{H})$.

A dichotomy (reflexive or transitive) behavior of subspaces of Toeplitz operators on the Hardy space on the unit disc was shown in [1]. There were presented conditions where the subspace is reflexive (not transitive) by existence of a Toeplitz operator in the predual with logarithmicaly summable symbol. Our aim is to find parallel characterization of subspaces of Toeplitz operators on the Hardy space on the upper half–plane.

The Hardy space $H^2(\mathbb{C}_+)$ on the upper half–plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is the space of all analytic functions $F : \mathbb{C}_+ \to \mathbb{C}$ such that

$$\|F\|_{H^2(\mathbb{C}_+)} := \sup_{y>0} \left(\int_{-\infty}^{\infty} |F(x+iy)|^2 dx \right)^{\frac{1}{2}} < \infty.$$

Since, for any function from $H^2(\mathbb{C}_+)$, there are limits a.e. from the upper half–plane to the real line, $H^2(\mathbb{C}_+)$ can be regarded as a subspace of $L^2(\mathbb{R})$ (see [2]). Let $P_{H^2(\mathbb{C}_+)}$ denote the orthogonal projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{C}_+)$.

For each $\Phi \in L^\infty(\mathbb{R})$ the *Toeplitz operator* with symbol Φ is an operator $T_\Phi \in \mathcal{B}(H^2(\mathbb{C}_+))$ defined by

$$T_\Phi F = P_{H^2(\mathbb{C}_+)}(\Phi F), \; F \in H^2(\mathbb{C}_+).$$

Let $\mathcal{T}(\mathbb{C}_+)$ denote the space of all Toeplitz operators.

The main result is

Theorem 1 *Suppose $\mathcal{F} \subset \mathcal{T}(\mathbb{C}_+)$ is a weak* closed subspace. Then the following statements are equivalent.*

1. *$\mathcal{F}$ is not transitive.*

2. *There is a function* $F : \mathbb{R} \to \mathbb{C}$ *such that* $F \in L^1(\mathbb{R})$, $\log|F| \in L^1\big(\mathbb{R}, \frac{dt}{1+t^2}\big)$ *and* $\int_{\mathbb{R}} \Phi F dt = 0$ *for all* $T_\Phi \in \mathcal{F}$.

3. $\mathcal{F}$ *is reflexive.*

Since the weak* topology plays important role in the proof the classical isomorphism between L^1 spaces on the real line and on the unit circle have to be redefined to get the proper relationship between weak* topology on the Toeplitz operators on the Hardy space on the upper half–plane and the Toeplitz operators on the Hardy space on the unit disc.

Joint work with M. Ptak.

References

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